MODULE* – 1

INTRODUCTION TO BEST APPROXIMATION AND FIXED POINT THEORY

Approximation theory is an old and rich branch of Analysis. The theory is as old as Mathematics itself. The ancient Greeks approximated the area of a closed curve by the area of a polygon. Since the particular examples of approximation often arise from problems of science and technology, they provide proper motivation for the subject of approximation theory.

The starting point of approximation theory is the concept of best approximation. Starting in 1853 , P.L. Chebyshev made significant contributions in the theory of best approximation. The problem of best approximation amounts to the problem of finding for a given point x and a given set G of a space X, a point $g_0 \in G$ which is nearest to x amongst all the points of the set G. Such an element g_0 , if it exists , is called a best approximation(or a nearest point or a closest point)to x in G.

For most of the available literature in the theory of best approximation , the underlying spaces are normed linear spaces (see e.g. [12],[15],[17],[24],[25],[41],[42],[58] and [65]).In more general spaces, results obtained do not constitute a unified theory as in the case of normed linear spaces. The construction of such a theory upto the present is an open problem although some attempts in this direction have been made by G.C.Ahuja , G.Albinus, E.W.Cheney, N.V.Efimov,T.D.Narang ,Geetha S.Rao, Ivan Singer , S.P.Singh ,S.B.Steckin, Swaran Trehan and many others (see e.g. [2],[4],[5],[15],[58],[60] and the references cited therein). One of our aim is also to make an attempt in this direction.

Fixed point theory plays an important role in functional analysis, differential equations, integeral equations, boundary value problems, statistics,engineering, economics etc. The problem of solving the equation f(x)=0 is equivalent to finding a fixed point of the mapping $y \rightarrow y-f(y)$. Since finding an exact solution of the equation f(x)=0 is not always possible, approximation theory comes to our rescue and we try to find an approximate solution (which is best possible subject to given constraints).Thus the two subjects of approximation theory and fixed point theory are closely related.

Applications of fixed point theorems to approximation theory are well known. Many results in approximation theory using fixed points are available in normed linear spaces.(see e.g. [6],[7].[17],[37],[61],[62]). Another aim in this module is to investigate applications of fixed point theorems to approximation theory when the underlying spaces are spaces more general than normed linear spaces.

To start with , we give a brief historical background of the subject, set up some notations, give few definitions contained in this module.

Throughout , R will denote the set of real numbers; R^+ will denote the set of non-negative real numbers; C^+, the set of complex numbers; iff for if and only if ; B(x,r) , a closed sphere with centre x and radius r ; ∂C, the boundary of a set C; X\E , the set of those points of X which are not in E; I ,the closed unit interval [0,1]; Card A, the number of infinite countable set and c, the cardinality of the interval.

Now we give a few definitions which frequently used for results in Best Approximation and Fixed Point Theory.

Definition 1.1 A subset G of a metric space (X,d) is said to be proximinal if for each $x \in X$ there exists a point g_0 in G which is nearest to x i.e.

$$d(x, g_0) = d(x,G) \equiv \inf\{d(x,g) : g \in G\} \qquad (1.1)$$

The term 'proximinal' was proposed by Raymond Killgrove (see Phelps [40], p. 790).

Every element $g_0 \in G$ satisfying (1.1) is called an element of best approximation of x by the elements of G or a nearest point or a closest point to x in G.

We shall denote by $P_G(x)$, the set of all best approximations to x in G i.e.

$$P_G(x) = \{g_0 \in G : d(x, g_0) = d(x, G)\}.$$

Thus G is proximinal if $P_G(x)$ is non-empty for each $x \in X$.

since

$$P_G(x) = \begin{cases} x, & x \in G \\ \varnothing, & x \in \overline{G} / G, \end{cases} \tag{1.2}$$

it follows that every proximinal set is closed.

The following example shows that a closed set need not be proximinal.

Example 1.1. Let $C_o = \{<a_n> : a_n \in F \ (F = R \text{ or } C),\ a_n \to 0\}$ with

$d(<a_n>, <b_n>) = \sup\limits_{n} d(a_n, b_n)$.

Let $M = \{<a_n> \in C_0 : \sum_{n \in N} 2_{an}^{-n} = 0\}$

then M is a closed infinite dimensional subset of C_o and if $x = <b_n> \notin M$, then there is no $m \notin M$ such that

$d(x,m) = d(x,M)$.

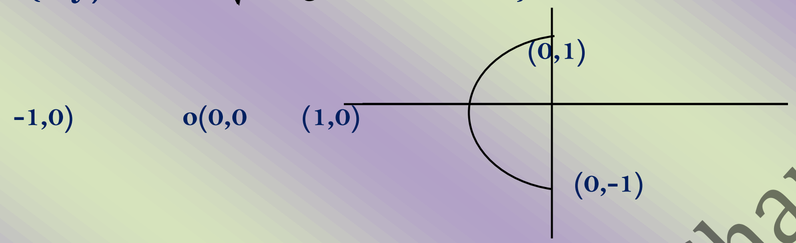
in view of (1.2), in order to exclude the trivial case when elements of best approximation do not exist, throughout while discussing $P_G(x)$ we shall assume, without special mention that $\overline{G} \neq X$.

In case $P_G(x)$ is exactly singleton (atmost singleton) for each $x \in X$, we have the following:

Definition 1.2. A set G in a metric space (X,d) is said to be Chebyshev or uniquely proximinal (semi-Chebyshev) If $P_G(x)$ consists of exactly one (atmost one) point for each x in X i.e. for each $x \in X$ there exists exactly one (atmost one) $g_0 \in G$ such that d(x, g_0)=d (x,G).

Example 1.2 [32]. Let X = R^2 with usual metric and

G = {x,y) : $x = -\sqrt{1-y^2}, -1 \le t \le 1\}$



If x=(1,0) then

$$P_G(x) = \{(0,1),(0,-1)\}$$

If x=(0,0) then $P_G(x)$=G.

The set G is proximinal but not Chebyshev.

Example 1.3[32] A closed bounded interval [a,b] on the real line is a Chebyshev set.

Definition1.3 The mapping which takes each point x of the space X to those points of the set G which are nearest to x is called a best approximation map or nearest point map or a metric projection.

Definition1.4 A set G in a metric space (X,d) is said to be approximatively compact (Effimov and Steckin [20]) if for every $x \in X$ and every sequence $<g_n>$ in G with

$$\lim_{n\to\infty} d(x,g_n) = d(x,G) \qquad (1.3)$$

there exists a sequence $<g_{n_i}>$ converging to an element of G. Any sequence $<g_n>$ satisfy (1.3) is called a minimizing sequence for x in G.

An approximatively compact set in a metric space is proximinal (Effimov and Steckin [20]) and since equation (1.2) implies that every proximinal set is closed,it follows that every proximinal compact set is closed, it follows that every approximatively compact set is closed. But a proximinal set need not be approximatively compact (Singer [59],p.389).

Definition1.5 A set G in a metric space (X,d) is said to be boundedly compact (Klee [30]) if every bounded sequence in G has a subsequence converging to a point of the space X. Equivalently, if the closure of G $\cap$ B is compact for each closed ball B in X,

In a metric space, every boundedly compact, closed set is approximatively compact (Effimov and Steckin [20]) and hence proximinal.

Definition1.6 Let (X,d) be a metric space and G a non-empty subset of X. An element $g_0 \in G$ is called a best co-approximation to x if

$$d(g_0, g) \leq d(x, g) \text{ for every } g \in G.$$

The set of all best co-approximation to $x \in X$ is denoted by $R_G(x)$.

Definition1.7 Let (X,d) be a metric space and G a non-empty subset of X. An element $g_0 \in G$ is an element of best simultaneous approximation (b.s.a.) to $x_1, x_2 \in X$ from G if

$$d(x_1, g_0) + d(x_2, g_0) = \inf\{d(x_1, g) + d(x_2, g) : g \in G\}$$

The set of all best simultaneous approximations to $x_1, x_2 \in X$ from G is denoted by $P_G(x_1, x_2)$.

Definition1.8 Let (X,d) be a metric space , G a non-empty subset of X and F a non-empty bounded subset of X. . An element $g_0 \in G$ is called an element of best simultaneous approximation of F with respect to G if

$$\sup_{y \in F} d(y, g_0) \leq \inf_{g \in G} \sup_{y \in F} d(y,g).$$

The set of all best simultaneous approximation to F with respect to G is denoted by $P_G(F)$.

Definition1.9 [10] An element x of a normed linear space X is said to be orthogonal to y $\in$X , $x \perp y$ if

$$d(x,0) \leq d(x, \alpha y)$$

for every scaiar α.

Correspondingly, we say that an element x of a metric linear space (X,d) is orthogonal to a subset M of X, if $x \perp y$ for each y in M.

Definition1.10 Let (X,d) be a metric space and C a subset of X. A mapping T:C------> X is said to be non-expansive if d(Tx,Ty) $\leq$d(x,y) for all x,y $\in$ C. The set F(T) = $\{x \in X : T(x) = x\}$is called the fixed point set of the mapping T and a point of F(T) is called a T-invariant point in X.

Definition1.11 For two non-empty sets A and B, a mapping

T:A--->B is called a retraction of A onto B if

(a) B is a subset of A,
(b) Tx = x for all x $\in$B.

The set B is said to be a retract (non-expansive retract) of A if there exists a retraction (n0on-expansive retraction) of A onto B.

Remark 1.1 The metric projection $\pi_G : X ---> G$ defined by $\pi_G(x) = \{g \in G : d(x,g) = d(x,G)\}$ is a retraction of X onto G.

Definition1.12 For a metric space (X,d) , a continuous mapping $W: X\times X\times I ---> X$ is said to be convex structure on X if for all x, y ∈ X , $\lambda \in I$,

$$d(u,W(x,y,\lambda))\leq \lambda d(u,x)+(1-\lambda)d(u,y)$$

for all u∈X. The metric space (X,d) together with a convex structure is called a convex metric space [64].

Clearly, a normed linear space or any convex subset of it is a convex metric space with W(x , y , λ) $=\lambda x+(1-\lambda)y$. But a linear metric space is not necessarily a convex metric space. There are many convex metric spaces (see Takashashi [64]) which cannot be embedded in any normed linear spaces. We give two preliminary examples here.

Example 1.4 [64]. Let I be the unit interval [0,1] and X be the family of closed intervals [a_i ,b_j] such that $0\leq a_i\leq b_j\leq 1$. For $I_i=[a_i,b_i]$, $I_j=[a_j,b_j]$ and $\lambda(0\leq\lambda\leq 1)$, we define a mapping W by W(I_i , I_j , λ) = [$\lambda a_i+(1-\lambda)a_j$, $\lambda b_i+(1-\lambda)b_j$] and define a metric d in X by the Hausdorff distance i.e.

$$d(I_i,I_j)=\sup_{a\in I}\{\left|\inf_{b\in I_i}(|a-b|),\inf_{c\in I_j}(|a-c|)\right|\}$$

Example 1.5 [64]. The linear space L which is also a metric space with the following properties:

(1) x, y ∈ L, d (x, y) = d (x-y, 0);

(2) For x, y ∈L and λ (0≤ λ ≤ 1),

$$d(\lambda x + (1-\lambda) y,0) \leq \lambda d (x, 0) + (1-\lambda) d (y, 0).$$

Definition1.13 A convex metric space (X,d) is said to be strictly convex [34] if for every x, y ∈X and r>0,d(x,p) ≤r, d(y,p) ≤r imply d(W(x, y, λ),p)<r unless x=y , where p is arbitrary but fixed point of X and $\lambda \in$ I.

Definition1.14 A non-empty subset C of a convex metric space (X,d) is said to be

(a) Starshaped [64] if there exists a p $\in$ C such that W(x ,p, λ) $\in$ C for every $\lambda \in$ I and for every x $\in$ C . Such a p is called a starcentre of C.

(b) Convex [64] , if W(x, y, λ) $\in$C whenever x,y $\in$ C and $\lambda \in$ I .

Clearly , a convex set is starshaped with respect to each of its points.

Definition1.15 A convex metric space (X,d) is said to satisfy property (I) [7], if for all x, y $\in$X and $\lambda \in$[0,1],

$$d(W(x,p,\lambda),W(y,p,\lambda)) \leq \lambda d(x,y) \qquad \text{(I)}$$

A convex metric space (X,d) is said to satisfy property (I*) if for all x, y $\in$X and $\lambda \in$[0,1],

$$d(W(x,p,\lambda),W(y,p,\lambda)) \leq (1-\lambda)d(x,y).$$

Clearly properties (I) and (I*) hold in normed linear spaces and in linear metric spaces satisfying

$$d(\lambda x+(1-\lambda)y,0) \leq \lambda d(x,0)+(1-\lambda)d(y,0). \qquad \text{(I*)}$$

Definition1.16 A normed linear space (X,$\|\cdot\|$) is said to be strictly convex if for any two points x and y of X and r> 0 with $\|x\| \leq r, \|y\| \leq r$ imply $\|(x+y)/2\|$< r unless x =y.

Definition1.17 A normed linear space (X,$\|\cdot\|$) is said to be pseudo strictly convex (P.S.C.) if given x $\neq$0 , y $\neq$0,$\|x+y\| = \|x\| + \|y\|$ implies y=tx for some t > 0.

For normed linear spaces , strict convexity and pseudo strictly convexity are equivalent (see e.g. [10] p. 122, [24] , [25] and [46]). Some authors use for such spaces the term strictly normed space or rotund space (see e.g. [16]).

Geometrically , strict convexity means that the spheres of the space contains no line segment on their surfaces. In such a space, if the sum of the lengths of two sides of a triangle is equal to the length of the third side, the triangle is degenerate. Three – dimensional strictly convex space is the one having a "football" shaped unit ball.

A very good account of strict convexity can be found in [27].

The nation of strict convexity was extended to metric linear spaces in [3] as under:-

Definition 1.18 A metric linear space (X,d) is said to the strictly convex if

$d(x,0) \leq r$,$d(y,0) \leq r$ imply $d(x+y)/2,0) < r$ unless $x=y$; $x,y \;\; X$ and r is any positive real number.

We now give a brief resume of the results contained in Best Approximation and Fixed Point Theory.

Best approximation and metric projections - To discuss some results on best approximation and metric projections, a notion best approximation in pseudo strictly convex metric linear spaces, was introduced and discussed by K.P.R. Sastry and S.V.R. Naidu in [45] and [46]. It was shown by Paul C. Kainen et al [28] that the existence of a continous best approximation in a strictly convex normed linear space X and taking values in a suitable subset M of X implies that M has the unique best approximation property this result of Paul C. Kainen at al was extended to pseudo strictly convex metric linear spaces by Sharma and Narang [53].

S.B. Steckin [54] proved that if $U_M = \{x \in X;$ Card $P_M(x) \leq 1\}$ then $U_M = X$ for every subset M of X iff X is a strictly convex normed linear metric space. This result was extended to strictly convex metric spaces by T.D. Narang [34]. A question that arises is what happens in spaces which are not strictly convex? To answer this , a characterization of multi-valued metric projection P_M in spaces which are not strictly convex along with the study of multivalued metric projections in convex metric linear spaces and convex metric spaces was discussed in [51] . For normed linear spaces which are not strictly convex, this result was proved by loan Serb in [47] . In the second section of [51] , it was proved by Sharma and Narang that for non-void proper subset M of a complete convex metric linear space X, P_M cannot be a countable multivalued metric projection. A characterizaton of the semi-metric linear spaces in terms of finitely-valued metric projections has also been discussed in this section. In [48], it was proved that if M is a strongly proximinal subset of a Banach Space X, then Card $P_M(x) \geq c$ for every xex\M, and the completeness of the space is essential for the validity of the result. In [48], the same result was proved for complete metrizable locally convex spaces i.e. in Frechet spaces. And in [51] it has been proved that for a strongly proximinal set M in a complete convex metric space (X,d), Card $P_M(x) \geq c$ for all x e X\M.

ε-Birkhoff Orthogonality and ε-Near Best Approximation - The nation of Birkhoff Orthogonalty, introduced in normed linear spaces in [9], was used to prove some results on best approximation (see [58], p.91). This notion of Orthgonality was extended to metric linear spaces by T.D. Narang and some results on best approximation were proved in [33]. A generalization of Birkhoff Orthogonality [9], called ϵ-Birkhoff

Orthogonality, was introduced by Sever Silvestru Dragomir [18] in normed linear spaces and this notion was used to prove a decomposition theorem ([18]-Theorem 3). We have extended This notion of ϵ-Birkhoff Orthogonality was extended and proved the decomposition theorem was proved in metric linear spaces by Sharma and Narang in [50] .

It was shown by Paul C. Kainen et al [28] that the existence of a continuous ϵ-near best approximation in a strictly convex normed linear space X and taking values in a suitable subset M implies that M has the unique best approximation property. By extending this result of Paul C. Kainen to convex metric spaces, it was proved in [50] that for a boundedly compact, closed subset M of a convex metric linear space (X ,d) which is also pseudo strictly convex, if for each $\varepsilon>0$, there exists a continuous e-near best approximation ø:X→M of X by M then M is a Chebyshev set. some other results on ε-near best approximation proved in [28] were extended to metric linear spaces in [50].

ε-Simultaneous Approximation and Best Simultaneous Co-approximation - The problem of best simultaneous approximation (b.s.a.) is concerned with approximating simultaneously elements x_1, x_2 of a metric space (X,d) by the elements of a subset G of X. More generally, if a set of elements B is given in X, one might like to approximate all the elements of B simultaneously by a single element of A. This type of problem arises when a function being approximated is not known precisely, but is known to belong to a set. C.B. Dunham [19] seems to be the first to have studied this problem of b.s.a. in normed linear spaces. The study was followed by J.B. Diaz and H.W. McLaughlin, W.H.Ling,

Goel at al and many others (see e.g. [1],[22],[36], [38],[39] and [44]). R.C. Buck [14] studied the problem of ϵ-approximation which reduces to the problem of best approximation for the particular case when ϵ=0. In the first section of this chapter, Defining ε simultaneous approximation map $P_G(\varepsilon)$:XxX→ 2^G (=the collection of all subsets of G) by $P_G(\varepsilon)(x_1,x_2)$ = {geG : d (x_1,go) +d (x_2,go)≤r+ϵ } where r = inf {d(x_1,g) + d (x_2,g) : geG} and $P_G(\varepsilon)$(F) = {g_0e G: $\sup_{y\in F}$ d (y, g_0) ≤ $\inf_{g\in G}$ $\sup_{y\in F}$ d (y,g) + e}, , the upper semi-continuity of the maps $P_G(\varepsilon)$(x_1, x_2) and $P_G(\varepsilon)$(F) and the convexity, boundedness, closedness and the starshpaedness of the sets $P_G(\varepsilon)$ (x_1, x_2) and $P_G(\varepsilon)$ (F) has been discussed by Sharma and Narang in [54] and [57] .

Best Simultaneous Co-approximation - This concept of best simultaneous co-approximation was introduced and discussed in normed linear spaces by c. Franchetti and M. Furi [21] in 1972. The study was taken up later by T.D. Narnag, P.L. Papini, Geetha S. Rao, Ivan Singer and few others (see [38],[39]. Generalizing the concept of best co-approxiamtion, Geetha S. Rao and R. Sarvanan studied the problem of best simultaneous co-approximation in nomed linear spaces in [43]. In the second section of [52] , Sharma and Narang have studied the problem of best simultaneous co-approximation in convex metric linear spaces and convex metric spaces, thereby extending some of the results proved in [43] and also by giving some properties of the set S_G (x,y) i.e. the set of all best simultaneous co-approxiamtions to x,y in G. It was also proved that for a convex metric space (X,d) G a convex subset of X and x, y eX, the set S_G (x,y) is a convex set. We have also proved the upper semi-continuity of the mapping S_G : {(x,y): x, yeX} $\to 2^G$ in totally complete metric linear spaces (a notion introduced By T.D. Narnag [35]).

Fixed points and approximation - The problem of fixed points of non-expansive mappings have been extensively discussed in strictly convex normed linear spaces (see e.g. [26]). It is known (see e.g. [10] Theorem 6, p.243) that for a closed convex subset K of a strictly convex normed linear space X and a non-expansive mapping T:K →X, the fixed point set (possibly empty) of T is a closed convex set. We have extended this result to pseudo strictly convex metric linear spaces.[53] Using fixed point theory, Brosowski [11] and Meinardus [31] established some interesting results on invariant approximation in normed linear spaces. Later various researchers obtained generalizations of their results (see e.g. [26] and the references cited therein). The object of the second section of [56] was to extend and generalize some results of Brosowski [11], Hicks and Humphries [23], Khan and Khan [29], and Singh [52], [53] in metric spaces having convex structure and in metric linear spaces having convex structure and in metric linear spaces having strictly monotone metric. Considering a subset C of a metric linear space with strictly monotone metric d and a non-expansive mapping T on $P_C(x)$ U{x} where x is a T-invariant point ,in [57] Sharma and Narang have proved the existence of an x_o in the set P_c (x) satisfying certain conditions , established a result on invariant approximation in strictly convex metric spaces and also have given an application of a fixed point theorem to ϵ-simultaneous approximation in convex metric spaces.

Non-expansive retracts in convex metric spaces - To generalize a theorem of Belluce and Kirk [8] on the existence of a common fixed point of a finite family of commuting non-expansive mappings, Ronald E.Bruck Jr. [13] studied some properties of fixed-point sets of non-expansive mappings in Banach spaces. In [55] , some of the results of [13] were extended to convex metric spaces it was also proved that the fixed point set of a non-expansive mapping satisfying conditional fixed property (CFP) is a non-expansive retract of C and hence metrically convex.

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